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# Note on Shape Theory (空間族における未解決問題)

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CITATION:

KODAMA, YUKIHIRO. Note on Shape Theory (空間族における未解決問題). 数理解析研究所講究録 1973, 194: 41-56

ISSUE DATE:

1973-12

URL:

<http://hdl.handle.net/2433/107283>

RIGHT:

# Note on shape theory

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## §1. Shape of compacta.

In [2], [3] K. Borsuk introduced the notion of shapes of metric compacta. Let  $X$  and  $Y$  be compacta lying in the Hilbert cube  $Q (= \prod_{n=1}^{\infty} I_n, I_n \text{ a copy of the interval } I = [0,1], n \in \mathbb{N}, \text{ where } \mathbb{N} \text{ is the set of positive integers})$ . A sequence  $\underline{f} = \{f_n\}$  of maps ( $=$  continuous maps)  $f_n : Q \rightarrow Q, n \in \mathbb{N}$ , is said to be a fundamental sequence of  $X$  to  $Y$  if for each neighborhood  $V$  of  $Y$  (in  $Q$ ) there is a neighborhood  $U$  of  $X$  such that  $f_n|_U \simeq f_{n+1}|_U$  in  $V$  for almost all  $n$ , that is, there is a homotopy  $H : U \times I \rightarrow V$  such that  $H(x,0) = f_n(x)$  and  $H(x,1) = f_{n+1}(x)$  for  $x \in U$ . We write  $\underline{f} : X \rightarrow Y$ . Setting  $i_n(x) = x$  for each  $x \in Q$ , for each compactum  $X \subset Q$   $\underline{i}_X = \{i_n\} : X \rightarrow X$  is a fundamental sequence which is called the fundamental identity sequence of  $X$ .

Two fundamental sequences  $\underline{f}, \underline{g} : X \rightarrow Y$  are said to be homotopic if for each neighborhood  $V$  of  $Y$  there is a neighborhood  $U$  of  $X$  such that  $f_n|_U \simeq g_n|_U$  in  $V$  for almost all  $n$ . We denote it by  $\underline{f} \simeq \underline{g}$ . The collection of all fundamental sequences ho-

motopic to a given fundamental sequence  $\underline{f}$  is said to be the fundamental class with the representative  $\underline{f}$  and it is denoted by  $[\underline{f}]$ .

The composition  $\underline{h} = \underline{g}\underline{f} : X \rightarrow Z$  of fundamental sequences  $\underline{f} : X \rightarrow Y$  and  $\underline{g} : Y \rightarrow Z$  is defined as the fundamental sequence consisting of maps  $h_n = g_n f_n : Q \rightarrow Q$ . If  $\underline{f} \simeq \underline{f}' : X \rightarrow Y$  and  $\underline{g} \simeq \underline{g}' : Y \rightarrow Z$ , then  $\underline{g}\underline{f} \simeq \underline{g}'\underline{f}' : X \rightarrow Z$ .

Each map  $f : X \rightarrow Y$  defines a fundamental sequence  $\underline{f} : X \rightarrow Y$  as follows. Take any extension  $h : Q \rightarrow Q$  of  $f$  and put  $f_n = h$  for each  $n$ . Then  $\underline{f} = \{f_n\}$  is a fundamental sequence of  $X$  to  $Y$ . We call  $\underline{f}$  the fundamental sequence induced by  $f$ .

Proposition 1. Let  $X$  and  $Y$  be 0-dimensional compacta. Then every fundamental sequence  $\underline{f} : X \rightarrow Y$  is induced by a map  $f : X \rightarrow Y$  and  $f$  is uniquely determined by  $\underline{f}$ .

Proof. Let  $\underline{f} = \{f_n\} : X \rightarrow Y$ . From the definition of a fundamental sequence and the compactness of  $Q$  the sequence  $\{f_n(x)\}$ ,  $x \in X$ , converges some point  $f(x)$  of  $Y$ . Obviously the correspondence  $x \rightarrow f(x)$ ,  $x \in X$ , defines a map  $f : X \rightarrow Y$  and it induces  $\underline{f}$ .

Compacta  $X$  and  $Y$  in  $Q$  are said to be fundamentally equivalent if there exist two fundamental sequences  $\underline{f} : X \rightarrow Y$  and  $\underline{g} : Y \rightarrow X$  such that  $\underline{g}\underline{f} \simeq \underline{i}_X$  and  $\underline{f}\underline{g} \simeq \underline{i}_Y$ . Then we write  $X \simeq_F Y$ . If we assume only that the relation  $\underline{g}\underline{f} \simeq \underline{i}_X$  holds, then we say that  $Y$  fundamentally dominates  $X$  and we write  $Y \geq_F X$ . If  $X$  and  $Y$  are homeomorphic, then  $X \simeq_F Y$ , because if  $\underline{f} : X \rightarrow Y$  and  $\underline{g} : Y \rightarrow X$  are fundamental sequences induced by  $f$  and  $g = f^{-1}$  then

$gf \simeq i_X$  and  $fg \simeq i_Y$ . Also, if  $X$  and  $Y$  are homotopically equivalent, then  $X \simeq_F Y$ .

It is known that the relation of the fundamental equivalence and the relation of the fundamental domination have an absolute character, that is, they do not depend on locations of compacta  $X$  and  $Y$  in  $Q$ . Since the relation of the fundamental equivalence is equivalence relation, the class of all compacta decomposes into mutually disjoint classes of compacta, called shapes. We denote by  $Sh(X)$  the class containing  $X$  and we call it the shape of  $X$ . Also we write  $Sh(X) \supseteq Sh(Y)$  if  $X \supseteq_F Y$ . If  $X$  and  $Y$  are ANR's (= compact ANR's for metric spaces), then it is known that  $Sh(X) \supseteq Sh(Y)$  if and only if  $X$  dominates homotopically  $Y$  and  $Sh(X) = Sh(Y)$  if and only if  $X$  and  $Y$  have same homotopy type. The shape of a space consisting of only one point is said to be trivial and denote by  $Sh(1)$ . If  $X$  is contractible, then it is obvious  $Sh(X) = Sh(1)$ .

Let  $X$  be a compactum contained in a compactum  $Y$ . A fundamental sequence  $\underline{f} = \{f_n\} : Y \rightarrow X$  is said to be a fundamental retraction if  $\underline{f}j_X \simeq i_X$ , where  $j_X : X \rightarrow Y$  is a fundamental sequence induced by the inclusion  $j_X : X \subset Y$ . If there is a fundamental retraction  $\underline{f} : Y \rightarrow X$ , then we call  $X$  a fundamental retract of  $Y$ . If there is a fundamental retraction  $\underline{f} : Y \rightarrow X$  such that  $\underline{f} \simeq i_Y$ , then  $X$  is a fundamental deformation retract of  $Y$ . A compactum  $X$  is said to be a fundamental absolute retract (a fundamental absolute neighborhood retract), written as FAR (FANR), if  $X$  is a fundamental retract of an AR (an ANR).

The following theorem characterizes a compactum with trivial shape ([4],[8],[10]).

Theorem 1. (K.Borsuk, D.M.Hyman, S.Mardešić) For a compactum X the followings are equivalent.

- (1) X is of trivial shape.
- (2) X is an FAR.
- (3) For a certain imbedding  $X \subset Q$  there is a sequence  $\{X_n\}$  of neighborhoods of X such that each  $X_n$  is homeomorphic to Q,  $X_{n+1} \subset \text{Interior } X_n$ ,  $n \in \mathbb{N}$ , and  $\bigcap_n X_n = X$ .

For a compactum X, Borsuk defined  $Fd(X)$ , the fundamental dimension of X, as the minimum of dimensions of all compacta Y with  $Sh(Y) \geq Sh(X)$ :

$$Fd(X) = \min_{Sh(Y) \geq Sh(X)} \dim Y$$

Obviously it holds that if  $Sh(X) \leq Sh(Y)$  then  $Fd(X) \leq Fd(Y)$  and if X and Y are compacta and  $Y \neq \emptyset$  then  $Fd(X) \leq Fd(X \times Y) \leq Fd(X) + Fd(Y)$ .

Let X be a compactum. A closed subset Y of X is said to be a fundamental k-skeleton of X if  $\dim Y \leq k$  and the homomorphisms  $\check{H}_n(Y;G) \rightarrow \check{H}_n(X;G)$  and  $\pi_n(Y, y_0) \rightarrow \pi_n(X, y_0)$  induced by the inclusion  $(Y, y_0) \subset (X, y_0)$ ,  $y_0$  a point of Y, are isomorphisms for  $0 \leq n < k$  and an epimorphism for  $n = k$ , where  $\check{H}_n(X;G)$  is the n-dimensional Čech homology group of X with coefficients in G and  $\pi_n(X, y_0)$  is the n-dimensional fundamental group of  $(X, y_0)$  defined by Borsuk [3].

We do not know whether every compactum has a fundamental 0-skeleton or not. If  $X$  is a solenoid of Van Dantzig, then  $X$  has a fundamental 0-skeleton which is homeomorphic to a Cantor discontinuum. (See Corollary of Theorem 5).

## §2. Approach to shapes by Mardešić and Segal.

By an ANR-sequence we imply an inverse sequence  $\underline{X} = \{X_n, \pi_{nn'}\}$  over the set of positive integers  $N$ , where  $X$  is an ANR and  $\pi_{nn'} : X_{n'} \rightarrow X_n$  is a map,  $n \in N$  ( $\pi_{nm} = \pi_{n+1, m} \cdots \pi_{n+1, n+1}$  for  $n < m$ ). Let  $\underline{X} = \varprojlim \underline{X}$  and let  $\pi_n : X \rightarrow X_n$  be the projection. A map  $\underline{f} : \underline{X} \rightarrow \underline{Y} = \{Y_n, \mu_{nn'}\}$  consists of an increasing function  $f : N \rightarrow N$  and of a collection of maps  $f_n : X_{f(n)} \rightarrow Y_n$  such that

$$f_n \pi_{f(n)f(n')} \simeq \mu_{nn'} f_{n'}, \quad \text{for } n \leq n', n, n' \in N.$$

Two maps  $\underline{f}, \underline{g} : \underline{X} \rightarrow \underline{Y}$  are said to be homotopic,  $\underline{f} \simeq \underline{g}$ , if for each  $n \in N$  there is an  $n' \in N$ ,  $n' \geq f(n), g(n)$ , such that

$$f_n \pi_{f(n)n'} \simeq g_n \pi_{g(n)n'}.$$

The composite  $\underline{gf} : \underline{X} \rightarrow \underline{Z}$  of  $\underline{f} : \underline{X} \rightarrow \underline{Y}$  and  $\underline{g} : \underline{Y} \rightarrow \underline{Z} = \{Z_n, \nu_{nn'}\}$  is a map of sequences  $\underline{h} : \underline{X} \rightarrow \underline{Z}$ , where  $h = fg : N \rightarrow N$  and  $h_n = g_n f_{g(n)} : X_{fg(n)} \rightarrow Z_n$ . The identity map of sequences  $\underline{i}_X : \underline{X} \rightarrow \underline{X}$  is given by the identity  $1_N : N \rightarrow N$  and the map  $i_{X_n} : X_n \rightarrow X_n$ ,  $n \in N$ . Two compacta  $X$  and  $Y$  are said to be of the same shape in the sense of ANR-systems, written as  $\overline{\text{Sh}}(X) = \overline{\text{Sh}}(Y)$ , provided there exist ANR-systems  $\underline{X}$  and  $\underline{Y}$  with  $X = \varprojlim \underline{X}$  and  $Y = \varprojlim \underline{Y}$  and maps  $\underline{f} : \underline{X} \rightarrow \underline{Y}$  and  $\underline{g} : \underline{Y} \rightarrow \underline{X}$  such that  $\underline{gf} \simeq \underline{i}_X$  and  $\underline{fg} \simeq \underline{i}_Y$ .

Mardešić and Segal [15, 16] gave the following useful cha-

racterization of shapes.

Theorem 2. (S.Mardešić and J.Segal) Let  $X$  and  $Y$  be compacta. Then  $\text{Sh}(X) = \text{Sh}(Y)$  if and only if  $\overline{\text{Sh}}(X) = \overline{\text{Sh}}(Y)$ .

## § 2. Shape of decomposition spaces.

According to Borsuk [4,p.266], a compactum  $X$  is said to be approximatively  $k$ -connected if for a certain imbedding  $X \subset Q$  and for every neighborhood  $V$  of  $X$  in  $Q$  there is a neighborhood  $U$  of  $X$  such that every map of a  $k$ -sphere  $S^k$  into  $U$  is null homotopic in  $V$ . It is known that the approximative  $k$ -connectedness is the shape invariant.

Theorem 3. (Kodama) Let  $f$  be a map of a compactum  $X$  onto a compactum  $Y$  with  $\dim Y \leq n$  such that for each  $y \in Y$   $f^{-1}(y)$  is approximatively  $k$ -connected,  $k = 0, 1, \dots, n$ . Then  $\text{Sh}(X) \geq \text{Sh}(Y)$ . Moreover, if  $\dim X \leq n$  then  $\text{Sh}(X) = \text{Sh}(Y)$ .

In the proof of Theorem 3 ([11]) an argument in the proof of Theorem of [9] is used essentially.

The following corollary is a generalization of Borsuk [4, Theorem (6.1)].

Corollary 1. An  $n$ -dimensional compactum  $X$  is of trivial shape if and only if  $X$  is approximatively  $k$ -connected for  $k = 0, 1, \dots, n$ .

For the proof it is sufficient to apply Theorem 3 to the case where  $Y$  is a space consisting of one point.

Corollary 2. (R.B.Sher) If  $X$  and  $Y$  are finite dimensional and  $f$  is a map of  $X$  onto  $Y$  such that  $f^{-1}(y)$  is of trivial shape

This is an immediate consequence of Theorems 1 and 3.

For a compactum  $X$ , denote by  $\square(X)$  the set of all components of  $X$ . We consider  $\square(X)$  as the decomposition space of  $X$ . Then it is a compactum. As an application of Theorem 3, we obtain the following theorem by Borsuk [3, Theorem (8.1)].

Corollary 3. (Borsuk) Let  $X, Y$  be compacta in  $Q$ . Then for every fundamental sequence  $f : X \rightarrow Y$  there is a unique (continuous) map  $\Lambda_f : \square X \rightarrow \square Y$  such that for each component  $X_0$  of  $X$   $f : X_0 \rightarrow \Lambda_f(X_0)$  is a fundamental sequence. Moreover  $\Lambda_f$  depends only on the fundamental class  $f$  and this dependence is covariant, that is, if  $g : Y \rightarrow Z$  is a fundamental sequence then  $\Lambda_{gf} = \Lambda_g \Lambda_f$ .

Proof. Let  $\pi_X : X \rightarrow \square X$  and  $\pi_Y : Y \rightarrow \square Y$  be the decomposition maps. Since  $\pi_X^{-1}(x)$  is a continuum for each  $x \in \square X$ , it is approximatively 0-connected. Since  $\dim \square X = 0$ , by Theorem 3 there is a fundamental sequence  $h : \square X \rightarrow X$  such that  $\pi_X h \simeq i_{\square X}$ . Consider  $\pi_Y f h : \square X \rightarrow \square Y$ . By Proposition 1,  $\pi_Y f h$  is induced by a map  $\Lambda_f : \square X \rightarrow \square Y$ . It is obvious that  $\Lambda_f$  satisfies Corollary 3.

The following generalizes Sher [21, Theorem 12] and it is given by a similar method as in a proof of Theorem 3 (cf. [11]).

Corollary 4. Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed compacta. Let  $f$  be a map of  $(X, x_0)$  onto  $(Y, y_0)$ . If  $f^{-1}(y)$  is approximatively  $k$ -connected for each  $y \in Y$  and  $k = 0, 1, \dots, n$ , then the induced homomorphism  $f_* : \pi_k(X, x_0) \rightarrow \pi_k(Y, y_0)$  is an isomorphism for  $k = 0, 1, \dots, n$ , where  $\pi_k$  is the  $k$ -dimensional fundamen-



tal group of Borsuk [3].

Corollary 5. Let  $f$  be a map of a compactum  $X$  onto an  $n$ -dimensional compactum  $Y$  such that  $f^{-1}(y)$  is approximately  $k$ -connected for each  $y \in Y$  and  $k = 0, 1, \dots, n$ . Then  $Fd(X) \geq Fd(Y)$ .

#### 4. $\Delta$ -spaces and fundamental dimension.

A compactum  $X$  is said to be a  $\Delta$ -space if there is an inverse sequence  $\{K_n, \pi_{n+1}\}$  of finite simplicial complexes such that  $X = \varprojlim \{K_n\}$  and each bonding map  $\pi_{n+1}: K_{n+1} \rightarrow K_n$  is simplicial.

Theorem 4. (Kodama) (1) Every 0-dimensional compactum and every finite polytope are  $\Delta$ -spaces.

(2) There is a 1-dimensional AR with property ( $\Delta$ ) which is not a  $\Delta$ -space.

(3) Every  $\Delta$ -space is dimensionally full-valued for paracompact spaces (cf. [14]).

In the shape category every compactum has a  $\Delta$ -space as its representative as shown by the following.

Theorem 5. (Kodama) For each compactum  $X$  there is a  $\Delta$ -space  $X'$  such that  $Sh(X) = Sh(X')$  and  $Fd(X) = Fd(X')$ .

Corollary. For every compactum  $X$  there is a compactum  $X'$  such that  $X'$  contains  $X$  as a fundamental deformation retract and  $X'$  has a fundamental  $k$ -skeleton for each  $k = 0, 1, 2, \dots$ .

We only give a proof of Corollary. For a given compactum  $X$ , find a  $\Delta$ -space  $Y$  by Theorem 5 such that  $Sh(X) = Sh(Y)$ . By Moszyńska [20] there is a compactum  $X'$  such that both  $X$  and  $Y$

are fundamental deformation retracts of  $X'$ . Let  $\{K_n, \pi_{n+1}\}$  be an inverse sequence of finite simplicial complexes such that  $Y = \varprojlim \{K_n\}$  and each  $\pi_{n+1}$  is simplicial. Denote by  $K_n^i$  the  $i$ -skeleton of  $K_n$ ,  $i = 0, 1, \dots$ . Then  $\{K_n^i, \pi_{n+1}\}$  forms an inverse sequence. Put  $Y_i = \varprojlim \{K_n^i\}$ ,  $i = 0, 1, \dots$ . Then it is obvious that  $Y_i$  is a fundamental  $i$ -skeleton of  $X'$ .

As shown in the above, every  $\Delta$ -space has a fundamental  $k$ -skeleton for each  $k = 0, 1, 2, \dots$ . On the other hand, consider the 2-dimensional continuum  $Q(\sigma)$  constructed in [10, p.390]. It is easy to know that  $Q(\sigma)$  has no fundamental 1-skeleton. Also, we can see that every ANR has a fundamental  $i$ -skeleton for  $i = 0, 1$ . The following example is a trivial modification of the example constructed by Borsuk [1].

Example. There is an infinite dimensional ANR  $X$  which does not have a fundamental  $k$ -skeleton for each  $k = 2, 3, \dots$ .

To find such an ANR  $X$ , let  $S^2$  be a 2-sphere and let  $A$  be an arc in  $S^2$ . Take a map  $f$  from  $A$  onto the Hilbert cube  $Q$  and let  $X$  be the adjunction space obtained by  $S^2$ ,  $Q$  and  $f$ . Then  $X$  is an infinite dimensional ANR. If  $X_k$  is a fundamental  $k$ -skeleton of  $X$  for  $k \geq 2$ , then  $X_k$  has to contain a subset  $S^2 - A$  of  $X$ . Since  $S^2 - A$  is dense in  $X$ , we have  $X_k = X$ . Thus there is no fundamental  $k$ -skeleton of  $X$ ,  $k = 2, 3, \dots$ .

Proposition 2. If  $X$  is a compactum in an  $n$ -dimensional euclidean space  $R^n$ , then  $Fd(X) \leq n - 1$ .

Proof. Take a sequence  $\{K_k\}$  of triangulable neighborhoods

of  $X$  in  $R^n$  such that  $K_{k+1} \subset K_k$ ,  $k = 1, 2, \dots$ , and  $\bigcap_k K_k = X$ . Since  $K_k$  is an  $n$ -dimensional polyhedron in  $R^n$ , there is a subpolyhedron  $L_k$  of  $K_k$  such that  $L_k$  is a strong deformation retract of  $K_k$  and  $\dim L_k \leq n-1$ . By induction, we can find a simplicial subdivision  $\tilde{L}_k$  of  $L_k$  and a simplicial map  $\pi_{k+1}: \tilde{L}_{k+1} \rightarrow \tilde{L}_k$  such that  $j_k \pi_{k+1} \simeq i_{k+1} j_{k+1} | \tilde{L}_{k+1}$  in  $K_k$  for  $k = 1, 2, \dots$ , where  $j_k: L_k \rightarrow K_k$ ,  $j_{k+1}: L_{k+1} \rightarrow K_{k+1}$  and  $i_{k+1}: K_{k+1} \rightarrow K_k$  are the inclusions. Consider the inverse sequence  $\{\tilde{L}_k, \pi_{k+1}\}$  and  $X' = \varprojlim \tilde{L}_k$ . It is known by Theorem 2 that  $\text{Sh}(X) = \text{Sh}(X')$ . Since  $\dim X' \leq n-1$ , we know  $\text{Fd}(X) \leq n-1$ .

Let  $\mathcal{C} = \{X_\alpha | \alpha \in \Lambda\}$  be a collection of compacta. A member  $X_0$  of  $\mathcal{C}$  is said to be majorant for the shapes of members of  $\mathcal{C}$  if  $\text{Sh}(X_0) \geq \text{Sh}(X_\alpha)$  for each  $X_\alpha \in \mathcal{C}$ . For example, let  $\mathcal{C}$  be the collection of all 0-dimensional compacta  $Y$  such that  $\text{Sh}(Y) \leq \text{Sh}(X)$  for a given compactum  $X$ . Then the decomposition space  $\square X$  of  $X$  consisting of all components of  $X$  is majorant for the shapes of members of  $\mathcal{C}$ . This follows from Corollary 3 of Theorem 3.

Proposition 3. (Watanabe) For the collection  $\mathcal{R}$  of all compacta in  $R^1$  a Cantor discontinuum is majorant for the shapes of members of  $\mathcal{R}$ .

This is a consequence of Proposition 2.

Theorem 6. (1) (S. Spieź) There is a compactum in  $R^2$  which is majorant for the shapes of all compacta in  $R^2$ .

(2) (K. Borsuk and W. Holsztyński) For the collection of

all solenoids  $\mathcal{D}$  no compactum  $X_0$  satisfies the condition  $\text{Sh}(X) \leq \text{Sh}(X_0)$  for every  $X \in \mathcal{D}$ . Hence, if  $\mathcal{C}$  is the collection of all compacta in  $\mathbb{R}^3$ , then there is no compactum which is majorant for the shapes of members of  $\mathcal{C}$ .

Problem 1. Let  $X$  be a compactum and let  $\mathcal{C}_X$  be the collection of all compacta  $Y$  such that  $\text{Sh}(X) > \text{Sh}(Y)$ . Does there exist a compactum which is majorant for the shapes of members of  $\mathcal{C}_X$ ?

The following problem is raised by Borsuk [3].

Problem 2. (Borsuk) Let  $X$  and  $Y$  be compacta. If  $\text{Fd}(Y) > 0$ , then does it hold  $\text{Fd}(X \times Y) \geq \text{Fd}(X) + 1$ ?

It is likely true that the following holds. However it does not know yet.

Problem 3. For every compactum  $X$ , does it hold that  $\text{Fd}(X \times S^1) = \text{Fd}(X) + 1$ ? Here  $S^1$  is a 1-sphere.

## §5. Movable compacta.

According to Borsuk [3,5], a compactum  $X$  in  $Q$  is said to be movable if for each neighborhood  $U$  of  $X$  there is a neighborhood  $V$  of  $X$  such that for every neighborhood  $W$  of  $X$  there is a homotopy  $H : V \times I \rightarrow U$  satisfying the condition:

$$H(x, 0) = x \quad \text{and} \quad H(x, 1) \in W \quad \text{for each } x \in V.$$

A compactum  $X$  is said to be k-movable if for every neighborhood  $U$  of  $X$  there is a neighborhood  $V$  of  $X$  such that for every compactum  $A \subset V$  with  $\dim A \leq k$  and for every neighborhood  $W$  of  $X$  there is a homotopy  $H : A \times I \rightarrow U$  satisfying the condition:

$$H(x,0) = x \quad \text{and} \quad H(x,1) \in W \quad \text{for } x \in A.$$

Mardešić and Segal [17] gave a characterization of movable compacta in terms of ANR sequences.

Theorem 7. (Mardešić and Segal) A compactum X is movable if and only if there is an ANR sequence  $\{X_n, \pi_{nn'}\}$  satisfying the following condition:  $X = \varprojlim \{X_n\}$  and for each  $n \in \mathbb{N}$  there is an  $n'$ ,  $n' \geq n$ , such that every  $n'' \geq n$  there is a map  $\mu_{n''n'}: X_{n'} \rightarrow X_{n''}$  satisfying the homotopy relation  $\mu_{n''n'} \pi_{nn''} \simeq \pi_{nn'}$ .

For movable compacta, the followings are known.

Theorem 8. (Borsuk) (1) Let X and Y be compacta with  $\text{Sh}(X) \geq \text{Sh}(Y)$ . If X is movable (k-movable), then Y is movable (k-movable).

(2) If X is movable (k-movable), then the suspension  $\Sigma X$  of X is movable (k-movable).

(3) Every compactum in  $\mathbb{R}^2$  is movable.

(4) If  $X_i$  is a movable compactum for  $i = 1, 2, \dots$ , then  $\prod_i X_i$  is movable.

(5) Every FANR is movable.

Theorem 9. (Kodama and Watanabe) An n-dimensional and n-movable compactum is movable.

Theorem 10. (1) (Mardešić) An n-dimensional  $LC^{n-1}$  compactum is movable.

(2) (Borsuk) An  $LC^{n-1}$  compactum is n-movable.

Let X be a  $\Delta$ -space. As we know from the proof of Corollary of Theorem 5, for each  $k = 0, 1, \dots$ , there is a fundamental k-skeleton  $X_k$  of X. It is easy to see  $X_k$  is i-movable

for  $i = 0, 1, \dots, k-1$ , if  $X$  is movable.

Problem 4. Let  $X$  be a movable  $\Delta$ -space. For each  $k = 1, 2, \dots$ , does there exist a fundamental  $k$ -skeleton  $X_k$  of  $X$  which is movable?

K. Borsuk [5] raised the following problems:

(1) Is it true that if  $X$  is  $m$ -movable and  $Y$  is  $n$ -movable then  $X \times Y$  is  $(m+n)$ -movable?

(2) Does there exist, for each  $n = 1, 2, \dots$ , a continuum which is  $n$ -movable, but is not  $(n+1)$ -movable?

(3) Does there exist a non-movable compactum which is  $n$ -movable for every  $n = 1, 2, \dots$ ?

These were solved by Kodama and Watanabe [12].

Theorem 11. (Kodama and Watanabe) (1) For compacta  $X$  and  $Y$ ,  $X \times Y$  is  $k$ -movable if and only if both  $X$  and  $Y$  are  $k$ -movable.

(2) If  $X$  is  $k$ -movable, then  $\Sigma X$  is  $(k+1)$ -movable.

(3) There is a continuum  $X$  such that  $X$  is  $k$ -movable for every  $k = 1, 2, \dots$ , but not movable.

To show (3) of Theorem 11, we remark that there is a non-movable continuum  $X_0$  such that  $\Sigma X_0$  is homeomorphic to  $X_0$  [7]. Since an  $n$ -fold suspension of a compactum  $X$  is  $(n-1)$ -movable by (2) of Theorem 11, the continuum  $X_0$  mentioned above is  $k$ -movable for every  $k = 1, 2, \dots$ . Borsuk proved every solenoid is not 1-movable. It is known that a suspension of a solenoid is 1-movable but not 2-movable.

It is known that every 2-dimensional ANR is dimensionally

full-valued (cf. [14]).

Problem 5. Is every 2-dimensional movable compactum dimensionally full-valued ?

Let  $X$  be a compactum with metric  $d$ . Let  $K$  be a finite simplicial complex and let  $V(K)$  be the set of vertices of  $K$ . For a map  $f : V(K) \rightarrow X$ , we mean by mesh  $f$  the maximum of diameters of  $f(s \cap V(K))$  for every simplex  $s$  of  $K$ . Let  $\epsilon > 0$ . For maps  $f, g : V(K) \rightarrow X$  with  $\max(\text{mesh } f, \text{mesh } g) < \epsilon$ , by  $f \sim_{\epsilon} g$  we imply that there is a sequence of maps  $h_i : V(K) \rightarrow X$ ,  $i = 0, 1, \dots, n$ , such that  $f = h_0$ ,  $g = h_n$ ,  $\text{mesh } h_i < \epsilon$ ,  $i = 0, 1, \dots, n$ , and  $\max\{d(h_i(v), h_{i+1}(v)) : v \in V(K)\} < \epsilon$ ,  $i = 0, 1, \dots, n-1$ .

Proposition 4. A compactum  $X$  is movable if and only if for every  $\epsilon > 0$  there is a  $\delta > 0$  satisfying the following conditions: For every finite simplicial complex  $K$ , every map  $f : V(K) \rightarrow X$  with mesh  $f < \delta$  and every  $\nu > 0$  there is a subdivision  $K'$  of  $K$  and a map  $g : V(K') \rightarrow X$  such that mesh  $g < \nu$  and  $f \sim_{\epsilon} g$ , where  $\pi : V(K') \rightarrow V(K)$  is a map defined by any projection of  $K'$  to  $K$ .

This proposition gives a simple proof of Theorem 10. In a similar form to Proposition 4 we can obtain a necessary and sufficient condition for a compactum  $X$  in order that  $X$  be an FANR.

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